

Approximate Integration Techniques

- Midpoint rule

$$\int_a^b f(x) dx \approx M_n = \Delta x [f(x_1) + f(x_2) + \dots + f(x_n)]$$

$$\Delta x = \frac{b-a}{n}$$

$$x_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

$$\text{Error: } Mn \leq \frac{M(b-a)^3}{24n^2}; M = |f''(\text{maximum})|$$

$$1. \int_0^3 4\sin(x) dx; n=3 \quad \Delta x = \frac{3-0}{3} = 1$$

$$x_i = \frac{1}{2}(x_{i-1} + x_i) \rightarrow x_1 = 0.5, x_2 = 1.5, x_3 = 2.5$$

$$M_3 = 4[\sin(0.5) + \sin(1.5) + \sin(2.5)] \approx 8.302$$

$$\text{error: } M_3 \leq \frac{3.99(3-0)^3}{24(3)^2}$$

$$M_3 \leq 0.499$$

$$M = |f''(\text{max})| = 3.99$$

$$f' = 4\cos x$$

$$f'' = -4\sin x$$

- Trapezoidal rule

$$\int_a^b f(x) dx \approx T_n = \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)]$$

$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$

$$\text{Error: } T_n \leq \frac{M(b-a)^3}{12n^2}; M = |f''(\text{maximum})|$$

$$2. \int_0^3 4 \sin(x) dx; n=3 \quad \Delta x = \frac{3-0}{3} = 1$$

$$x_i: 0+i(1) = i \rightarrow x_0=0, x_1=1, x_2=2, x_3=3$$

$$T_3 = \frac{1}{2} [4 \sin(0) + 2(4 \sin(1)) + 2(4 \sin(2)) + 4 \sin(3)] = 7.285$$

$$\text{error: } T_3 \leq \frac{3.99(3-0)^3}{12(3)^2}$$

$$T_3 \leq 0.998$$

$$M = |f''|_{\max} = 3.99$$

$$f' = 4 \cos x$$

$$f'' = -4 \sin x$$

- Simpson's rule

$$\int_a^b f(x) dx \approx S_n = \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

$$\Delta x = \frac{b-a}{n}$$

$$\text{Error: } S_n \leq \frac{M(b-a)^5}{180n^4}; M \geq |f^{(4)}(x)|$$

$$3. \int_0^4 4 \sin(x) dx; n=4 \quad \Delta x = \frac{4-0}{4} = 1$$

$$x_0=0, x_1=1, x_2=2, x_3=3, x_4=4$$

$$S_4 = \frac{1}{3} [4 \sin(0) + 4(4 \sin(1)) + 2(4 \sin(2)) + 4(4 \sin(3)) + 4 \sin(4)] = 6.656$$

$$\text{error: } S_4 \leq \frac{\frac{1}{4}(4-0)^5}{180(4)^4}$$

$$S_4 \leq 0.001$$

$$M \geq |f^{(4)}(x)|$$

$$f' = 4 \cos x$$

$$f'' = -4 \sin x$$

$$f''' = -4 \cos x$$

$$f^{(4)} = |4 \sin x| \leq 1 \Rightarrow |\sin x| \leq \frac{1}{4}$$