Test Prep 5 Worksheet 1

1. $x = 7\sin(y), 0 \leq y \leq \frac{\pi}{2}$, set up the integral to find the length of the curve

$x' = 7\cos(y) \qquad L = \int_{y_1}^{y_2} \sqrt{1 + (x')^2}\, dy$

$L = \int_0^{\pi/2} \sqrt{1 + (7\cos(y))^2}\, dy = \int_0^{\pi/2} \sqrt{1 + 49\cos^2(y)}\, dy$

2. $y = \frac{1}{4}x^2 - \frac{1}{2}\ln(x), 1 \leq x \leq 8$, find the exact length of the curve

$y' = \frac{1}{2}x - \frac{1}{2x} \qquad L = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx$

$1 + (y')^2 = 1 + \left(\frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2}\right) = \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{1}{2}x + \frac{1}{2x}\right)^2$

$L = \int_1^8 \sqrt{\left(\frac{1}{2}x + \frac{1}{2x}\right)^2}\, dx = \int_1^8 \left|\frac{1}{2}x + \frac{1}{2x}\right| dx = \left[\frac{1}{4}x^2 + \frac{1}{2}\ln(x)\right]_1^8 = \left(\frac{1}{4}(8)^2 + \frac{1}{2}\ln(8)\right) - \left(\frac{1}{4} + \frac{1}{2}\ln(1)\right)$

$= \frac{63}{4} + \frac{1}{2}\ln(8)$

3. $y = \frac{1}{2}x^2, P\left(-5, \frac{25}{2}\right), Q\left(5, \frac{25}{2}\right)$, set up the integral in terms of x, then find the arc length from P to Q

$y' = x \qquad L = \int_{P(x)}^{Q(x)} \sqrt{1 + (y')^2}\, dx$

$L = \int_{-5}^{5} \sqrt{1 + x^2}\, dx = 2\int_0^5 \sqrt{1 + x^2}\, dx = 2\left[\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\ln\left(x + \sqrt{1+x^2}\right)\right]_0^5$

$= 2\left[\left(\frac{5}{2}\sqrt{1+5^2} + \frac{1}{2}\ln\left(5 + \sqrt{1+5^2}\right)\right) - \left(0 + \frac{1}{2}\ln\sqrt{1}\right)\right] = 5\sqrt{26} + \ln\left(5 + \sqrt{26}\right)$

4. $x = \ln(8y+1), 0 \le y \le 1$, integrate with respect to x and with respect to y

$x' = \frac{8}{8y+1}$

$e^x = 8y+1 \Rightarrow y = \frac{1}{8}e^x - \frac{1}{8} \qquad 0 \le x \le \ln(9) \qquad y' = \frac{1}{8}e^x$

$S = \int_{x_1}^{x_2} 2\pi y\sqrt{1+(y')^2}\,dx \quad + \quad S = \int_{y_1}^{y_2} 2\pi x\sqrt{1+(x')^2}\,dy$

$S_x = \int_0^{\ln(9)} 2\pi\left(\frac{1}{8}e^x - \frac{1}{8}\right)\sqrt{1+\left(\frac{1}{8}e^x\right)^2}\,dx = \int_0^{\ln(9)} \pi\left(\frac{1}{4}e^x - \frac{1}{4}\right)\sqrt{1+\frac{1}{64}e^{2x}}\,dx$

$S_y = \int_0^1 2\pi(\ln(8y+1))\sqrt{1+\left(\frac{8}{8y+1}\right)^2}\,dy = \int_0^1 2\pi(\ln(8y+1))\sqrt{1+\frac{64}{(8y+1)^2}}\,dy$

5. $y = \cos\left(\frac{1}{4}x\right), 0 \le x \le 2\pi$, find the exact area of the surface rotating about the x-axis

$y' = -\frac{1}{4}\sin\left(\frac{1}{4}x\right) \qquad S = \int_{x_1}^{x_2} 2\pi y\sqrt{1+(y')^2}\,dx$

$S = \int_0^{2\pi} 2\pi\left(\cos\left(\frac{1}{4}x\right)\right)\sqrt{1+\left(-\frac{1}{4}\sin\left(\frac{1}{4}x\right)\right)^2}\,dx = 2\pi\int_0^{2\pi} \cos\left(\frac{1}{4}x\right)\sqrt{1+\frac{1}{16}\sin^2\left(\frac{1}{4}x\right)}\,dx$

$u = \sin\left(\frac{1}{4}x\right) \to \sin(0) = 0, \ \sin\left(\frac{\pi}{2}\right) = 1 \qquad 0 \le u \le 1$

$du = \frac{1}{4}\cos\left(\frac{1}{4}x\right)$

$= 2\pi\int_0^1 4\sqrt{1+\frac{1}{16}u^2}\,du = 2\pi\int_0^1 \sqrt{16+u^2}\,du = 2\pi\left[\frac{u}{2}\sqrt{16+u^2} + 8\ln\left(u+\sqrt{16+u^2}\right)\right]_0^1$

$= 2\pi\left[\left(\frac{1}{2}\sqrt{17} + 8\ln(1+\sqrt{17})\right) - (0 + 8\ln(4))\right] = \pi\sqrt{17} + 16\pi\ln\left(\frac{1+\sqrt{17}}{4}\right)$

6. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4, 0 \le y \le 8$, find the area of the surface rotation about the y-axis

$y^{2/3} = 4 - x^{2/3} \rightarrow y = (4-x^{2/3})^{3/2} \qquad y' = \frac{3}{2}(4-x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = -\frac{\sqrt{4-x^{2/3}}}{x^{1/3}}$

$1+(y')^2 = 1 + \frac{4-x^{2/3}}{x^{2/3}} = \frac{x^{2/3}+4-x^{2/3}}{x^{2/3}} = 4x^{-2/3} \qquad S = \int_0^8 2\pi y\sqrt{1+(y')^2}\,dx = 2\pi\int_0^8 (4-x^{2/3})^{3/2}\sqrt{4x^{-2/3}}\,dx$

$= 2\pi\int_0^8 (8-x)(2x^{-1/3})\,dx = 2\pi\int_0^8 \frac{16}{x^{1/3}} - 2x^{2/3}\,dx = 2\pi\left[24x^{2/3} - \frac{6}{5}x^{5/3}\right]_0^8 = 2\pi\left[24(8)^{2/3} - \frac{6}{5}(8)^{5/3}\right]$

$= 2\pi\left(96 - \frac{192}{5}\right) = \frac{576\pi}{5}$