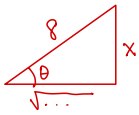
Exam 2 Prep- Worksheet 2

- Trig subs
  - $\int \sqrt{a^2 - x^2}\, dx$
    - $\int \sqrt{2304 - 36x^2} dx$

$\int 6\sqrt{8^2 - x^2}\, dx$

$a = 8$ $\quad x = 8\sin\theta$ $\quad \sin\theta = \frac{x}{8}$

$dx = 8\cos\theta\, d\theta$ $\quad \theta = \sin^{-1}\frac{x}{8}$



$6\int 8\sqrt{\sin^2\theta - 1} \cdot 8\cos\theta\, d\theta = 384\int \cos^2\theta\, d\theta = 384\int \frac{1+\cos(2\theta)}{2} d\theta$

$= 192\int 1 + \cos(2\theta)\, d\theta = 192\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C = 192\left[\sin^{-1}\frac{x}{8} + \frac{1}{2}(2\sin\theta\cos\theta)\right] + C$

$= 192\left[\sin^{-1}\frac{x}{8} + \left(\frac{x}{8} \cdot \frac{\sqrt{64 - x^2}}{8}\right)\right] + C$

  - $\int \sqrt{x^2 - a^2}\, dx$
    - $\int \sqrt{16x^2 - 400} dx$

$\int 4\sqrt{x^2 - 25}\, dx$

$a = 5$ $\quad x = 5\sec\theta$ $\quad \sec\theta = \frac{x}{5}$

$dx = 5\sec\theta\tan\theta\, d\theta$ $\quad \theta = \sec^{-1}\frac{x}{5}$

$4\int \sqrt{(5\sec\theta)^2 - 25} \cdot 5\sec\theta\tan\theta\, d\theta = 4\int \sqrt{25(\sec^2\theta - 1)} \cdot 5\sec\theta\tan\theta\, d\theta$

$4\int 5\sqrt{\tan^2\theta} \cdot 5\sec\theta\tan\theta = 100\int \tan\theta\sec\theta\tan\theta\, d\theta = 100\int \sec\theta\tan^2\theta\, d\theta$

$100\int \sec\theta(\sec^2\theta - 1)\, d\theta = 100\int \sec^3\theta - \sec\theta\, d\theta = 100\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\, d\theta - \int \sec\theta\, d\theta\right]$

$100\left[\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\int \sec\theta\, d\theta\right] = 50\left[\sec\theta\tan\theta - \int \sec\theta\, d\theta\right] = 50\left[\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|\right] + C$

$50\left[\frac{x}{5} \cdot \frac{\sqrt{16x^2 - 400}}{5} - \ln\left|\frac{x}{5} + \frac{\sqrt{16x^2 - 400}}{5}\right|\right] + C$

- $\int \sqrt{x^2 + a^2}\, dx$
  - $\int \sqrt{81x^2 + 5184} dx$

$\int 9\sqrt{x^2+64}$

$a = 8$ $\quad x = 8\tan\theta$ $\quad \tan\theta = \frac{x}{8}$

$dx = 8\sec^2\theta$ $\quad \theta = \tan^{-1}\frac{x}{8}$

$9\int\sqrt{(8\tan\theta)^2+64} \cdot 8\sec^2\theta\, d\theta = 9\int 8\sqrt{\tan^2\theta+1} \cdot 8\sec^2\theta\, d\theta$

$9\int 8\sqrt{\sec^2\theta} \cdot 8\sec^2\theta\, d\theta = 576\int\sec^3\theta\, d\theta = 576\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int\sec\theta\, d\theta\right]$

$288\left[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right]+c = 288\left[\frac{\sqrt{81x^2+5184}}{8}\cdot\frac{x}{8} + \ln\left|\frac{\sqrt{81x^2+5184}}{8} + \frac{x}{8}\right|\right]+c$

- Partial fraction decomposition
  - Repeated linear
    - $\int \frac{3x+9}{x^2-4x-5} dx = \int \frac{3x+9}{(x-5)(x+1)} dx$ $\qquad \frac{3x+9}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x+1} \rightarrow 3x+9 = A(x+1)+B(x-5)$

$x=-1 \rightarrow 3(-1)+9 = B(-1-5) \rightarrow B=-1$

$x=5 \rightarrow 3(5)+9 = A(5+1) \rightarrow A=4$

$\int \frac{4}{x-5} - \frac{1}{x+1} dx = 4\int\frac{1}{x-5}dx - \int\frac{1}{x+1}dx$

$4\ln|x-5| - \ln|x+1| + c$

- Non-repeated quadratic
  - $\int \frac{1}{(x-5)(x^2+1)}dx$

$$\frac{1}{(x-5)(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{D}{x-5}$$

$1 = (Ax+B)(x-5)+D(x^2+1)$

$1 = Ax^2 - 5Ax + Bx - 5B + Dx^2 + D$

$1 = (A+D)x^2 + (-5A+B)x + (-5B+D)$

$A+D=0 \rightarrow A=-D$ $\boxed{A=-\frac{1}{26}}$

$-5A+B=0 \rightarrow 5D+B=0 \rightarrow B=-5D$ $\boxed{B=-\frac{5}{26}}$

$-5B+D=1 \rightarrow 25D+D=1 \rightarrow 26D=1 \rightarrow$ $\boxed{D=\frac{1}{26}}$

$$\int \frac{-\frac{1}{26}x-\frac{5}{26}}{x^2+1} + \frac{\frac{1}{26}}{x-5}dx = -\frac{1}{26}\int\frac{x+5}{x^2+1}dx + \frac{1}{26}\int\frac{1}{x-5}dx = -\frac{1}{26}\left[\int\frac{x}{x^2+1}dx + 5\int\frac{1}{x^2+1}dx\right] + \frac{1}{26}\int\frac{1}{x-5}dx$$

$u=x^2+1$
$du=2xdx$

$$-\frac{1}{26}\left[\frac{1}{2}\int\frac{1}{u}du + 5\int\frac{1}{x^2+1}dx\right] + \frac{1}{26}\int\frac{1}{x-5}dx = -\frac{1}{26}\left[\frac{1}{2}\ln|u| + 5\tan^{-1}(x)\right] + \frac{1}{26}\ln|x-5| + C$$

$$-\frac{1}{26}\left[\frac{1}{2}\ln|x^2+1| + 5\tan^{-1}(x)\right] + \frac{1}{26}\ln|x-5| + C$$

- Repeated quadratic
  - $\int \frac{x}{(x+1)(x^2+1)^2}dx$

$$\frac{x}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$$

x^4+2x^2+1 $\quad$ x^3-x^2+x-1

$x = A(x^2+1)^2 + (Bx+D)(x-1)(x^2+1) + (Ex+F)(x-1)$

$x = Ax^4 + 2Ax^2 + A + Bx^4 - Bx^3 + Bx^2 - Bx + Dx^3 - Dx^2 + Dx - D + Ex^2 - Ex + Fx - F$

$x = (A+B)x^4 + (-B+D)x^3 + (2A+B-D+E)x^2 + (-B+D-E+F)x + (A-D-F)$

$A+B = 0$
$-B+D = 0$
$2A+B-D+E = 0$
$-B+D-E+F = 1$
$A - D - F = 0$

$B=D=-A$
$F=2A$
$E=-2A$

$\cancel{A}\ A+2A+2A=1$
$4A=1 \rightarrow \boxed{A=\frac{1}{4}}$

$\boxed{B=-\frac{1}{4}}$ $\boxed{D=-\frac{1}{4}}$ $\boxed{E=-\frac{1}{2}}$ $\boxed{F=\frac{1}{2}}$

$$\int \frac{\frac{1}{4}}{x-1} + \frac{-\frac{1}{4}x+(-\frac{1}{4})}{x^2+1} + \frac{-\frac{1}{2}x+\frac{1}{2}}{(x^2+1)^2}dx$$

$u=x^2+1$ $\quad$ $du=2xdx$

$$\frac{1}{4}\int\frac{1}{x-1}dx - \frac{1}{4}\int\frac{x+1}{x^2+1}dx - \frac{1}{2}\int\frac{x-1}{(x^2+1)^2}dx = \frac{1}{4}\ln|x-1| - \frac{1}{4}\left[\frac{1}{2}\ln|x^2+1| + \tan^{-1}x\right] - \frac{1}{2}\left[-\frac{1}{2}\frac{1}{x^2+1} - \frac{1}{2}\left(\tan^{-1}x + \frac{x}{\sqrt{x^2+1}}\cdot\frac{1}{\sqrt{x^2+1}}\right)\right]$$

$$\frac{1}{4}\ln|x-1| - \frac{1}{8}\left(\ln|x^2+1| + \tan^{-1}x\right) + \frac{1}{4}\left(\frac{1}{x^2+1} + \tan^{-1}x + \frac{x}{x^2+1}\right) + C$$

- Completing the square
  - $\int \frac{x}{(x+3)(x^2+4x+5)}dx = \int \frac{x}{(x+3)[(x+2)^2+1]}dx$

$u = x+2 \rightarrow x = u-2$

$du = dx$

$\int \frac{u-2}{[(u-2)+3][u^2+1]}du = \int \frac{u-2}{(u+1)(u^2+1)}du$

$\frac{u-2}{(u+1)(u^2+1)} = \frac{A}{u+1} + \frac{Bu+D}{u^2+1} \rightarrow u-2 = A(u^2+1) + (Bu+D)(u+1)$

$u-2 = Au^2 + A + Bu^2 + Bu + Du + D$

$u-2 = (A+B)u^2 + (B+D)u + (A+D)$

$A+B = 0 \rightarrow B = -A$

$B+D = 1$ $\quad -A+D = 1$

$A + D = -2$ $\quad A+D = -2$

$2D = -1 \rightarrow$

$A = -\frac{3}{2}$
$B = \frac{3}{2}$
$D = -\frac{1}{2}$

$-\frac{3}{2}\int \frac{1}{u+1}du + \frac{1}{2}\int \frac{3u-1}{u^2+1}du = -\frac{3}{2}\ln|u+1| + \frac{1}{2}\left[3\int \frac{u}{u^2+1}du - \int \frac{1}{u^2+1}du\right]$

$w = u^2+1$

$dw = 2u\,du$

$-\frac{3}{2}\ln|u+1| + \frac{1}{2}\left[\frac{3}{2}\int \frac{1}{w}dw - \tan^{-1}u\right] = -\frac{3}{2}\ln|u+1| + \frac{1}{2}\left[\frac{3}{2}\ln|w| - \tan^{-1}u\right] + C$

$-\frac{3}{2}\ln|u+1| + \frac{1}{2}\left[\frac{3}{2}\ln|u^2+1| - \tan^{-1}u\right] + C = -\frac{3}{2}\ln|x+3| + \frac{3}{4}\ln|(x+2)^2+1| - \frac{1}{2}\tan^{-1}(x+2) + C$

- Numerical integration
  - Midpoint rule
    - $\int_0^5 4x^5 + 2x^2 + 2x\,dx$; n=5

$\Delta x = \frac{5-0}{5} = 1$ $\quad x_i = \frac{1}{2}(x_{i-1} + x_i)$

$x_1 = 0.5,\ x_2 = 1.5,\ x_3 = 2.5,\ x_4 = 3.5,\ x_5 = 4.5$

$M_5 = [4(0.5)^5 + 2(0.5)^2 + 2(0.5)] + [4(1.5)^5 + 2(1.5)^2 + 2(1.5)] + [4(2.5)^5 + 2(2.5)^2 + 2(2.5)] + [4(3.5)^5 + 2(3.5)^2 + 2(3.5)]$

$+ [4(4.5)^5 + 2(4.5)^2 + 2(4.5)] \approx 10010.625$

- Trapezoidal rule
  - $\int_0^5 4x^5 + 2x^2 + 2x dx$; n=5

$\Delta x = 1$ $x_i = i$

$x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5$

$$T_5 = \frac{1}{2}\left\{[4(0)^5+2(0)^2+2(0)]+[4(1)^5+2(1)^2+2(1)]+[4(2)^5+2(2)^2+2(2)]+[4(3)^5+2(3)^2+2(3)]+[4(4)^5+2(4)^2+2(4)]+[4(5)^5+2(5)^2+2(5)]\right\}$$

≈ 11560

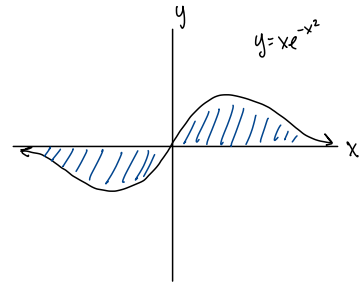
- Simpson's rule
  - $\int_0^5 4x^5 + 2x^2 + 2x dx$; n=5

$\Delta x = 1$ $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5$

$$S_5 = \frac{1}{3}\left\{[4(0)^5+2(0)^2+2(0)]+[4(1)^5+2(1)^2+2(1)]+[4(2)^5+2(2)^2+2(2)]+[4(3)^5+2(3)^2+2(3)]+[4(4)^5+2(4)^2+2(4)]+[4(5)^5+2(5)^2+2(5)]\right.$$

≈ 25122.667

- Improper integrals
  - Infinite intervals
    - $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$



$$\int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx$$

$$\lim_{a\to-\infty}\int_{a}^{0} xe^{-x^2}dx + \lim_{b\to\infty}\int_{0}^{b} xe^{-x^2}dx$$

$$\lim_{a\to-\infty}\left[-\frac{1}{2}e^{-x^2}\right]_{a}^{0} + \lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_{0}^{b}$$

$$\lim_{a\to-\infty}\left[-\frac{1}{2}-\left(-\frac{1}{2}e^{-a^2}\right)\right] + \lim_{b\to\infty}\left[-\frac{1}{2}e^{-b^2}-\left(-\frac{1}{2}\right)\right]$$

$$\left[-\frac{1}{2}+0\right]+\left[0+\frac{1}{2}\right] = 0$$

$$\int xe^{-x^2}dx \qquad u=-x^2$$

$$\int e^{u}\left(-\frac{1}{2}\right)du \qquad du=-2x\,dx$$

$$=-\frac{1}{2}e^{u}+C \to -\frac{1}{2}e^{-x^2}+C$$

  - Discontinuous intervals
    - $\int_{1}^{5} \frac{1}{\sqrt{x-1}}\,dx$

$$\lim_{a\to1^+}\int_{a}^{5}(x-1)^{-1/2}dx = \lim_{a\to1^+}\left.\frac{(x-1)^{1/2}}{1/2}\right|_{a}^{5} = \lim_{a\to1^+}\left.2(x-1)^{1/2}\right|_{a}^{5}$$

$$2\lim_{a\to1^+}\left[4^{1/2}-(a-1)^{1/2}\right] = 2\left[2-0^{1/2}\right] = 4$$