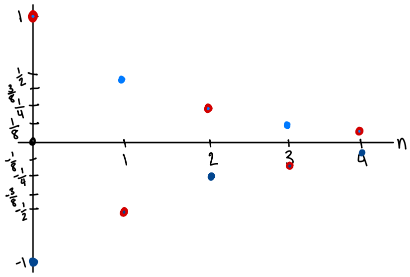
Test Prep 3 Worksheet 1

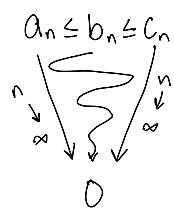
1. $a_n = -\frac{(-8)^n}{3}$

$a_1 = \frac{8}{3}$, $a_2 = -\frac{64}{3}$, $a_3 = \frac{512}{3}$, $a_4 = -\frac{4096}{3}$

2. $\lim_{n\to\infty} \frac{(-1)^n}{2^n}$

- $b_n = \left[\frac{(-1)^n}{2^n}\right]_{n=0}^{\infty} \Rightarrow \left[1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \dots\right]$
- $a_n = \left[-\frac{1}{2^n}\right]_{n=0}^{\infty} \Rightarrow \left[-1, -\frac{1}{2}, -\frac{1}{4}, \dots\right]$
- $c_n = \left[\frac{1}{2^n}\right]_{n=0}^{\infty} \Rightarrow \left[1, \frac{1}{2}, \frac{1}{4}, \dots\right]$





$$\left.\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} -\frac{1}{2^n} = 0 =: L \\ \lim_{n\to\infty} c_n &= \lim_{n\to\infty} \frac{1}{2^n} = 0 =: L\end{aligned}\right\} \text{agree!}$$

Squeeze theorem for sequence indicates $\lim_{n\to\infty} \left[\frac{(-1)^n}{2^n}\right] = L =: 0$

$\lim_{n\to\infty} |b_n| = \lim_{n\to\infty} \left|\frac{(-1)^n}{2^n}\right| = \lim_{n\to\infty} \frac{1}{2^n} = 0$ ← check ✓

3. $\sum_{k=1}^{\infty} \left(3\left(\frac{1}{2}\right)^{k-1} + 5\left(\frac{1}{3}\right)^{k}\right)$

$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$

$$\underbrace{\sum_{k=1}^{\infty} 3\left(\frac{1}{2}\right)^{k-1}} + \sum_{k=1}^{\infty} 5\left(\frac{1}{3}\right)^{k}$$

$$\frac{3}{1-\frac{1}{2}} + \underbrace{\sum_{k=1}^{\infty} 5 \cdot \frac{1}{3}\left(\frac{1}{3}\right)^{k-1}}$$

$$\frac{3}{\frac{1}{2}} + \frac{5/3}{1-\frac{1}{3}} = 6 + \frac{5}{3}\left(\frac{1}{2/3}\right) = 6 + \frac{5}{3}\left(\frac{3}{2}\right) = \frac{17}{2}$$

harmonic series: diverges

4. $\lim_{n=1} \frac{3^n}{7n^2-5} = \frac{\infty}{\infty}$

$\stackrel{LH}{=} \lim_{n\to\infty} \frac{3^n \ln(3)}{14n} = \frac{\infty}{\infty}$

$\stackrel{LH}{=} \lim_{n\to\infty} \frac{3^n \ln^2(3)}{14} = \frac{\infty}{14} = \infty \neq 0 \rightarrow$ diverges

5. $\sum_{n=1}^{\infty} \frac{1}{n+1}$

1) f is cont. $x > -1$ $N=1$ $x \geq N=1$

2) $f(x) = \frac{1}{x+1}$

$f'(x) = -\frac{1}{(x+1)^2} < 0, \quad x \geq N=1$

3) $a_n = f(n)$

$\int_1^\infty \frac{1}{x+1}\,dx = \lim_{k\to\infty} \int_1^k \frac{1}{x+1}\,dx = \lim_{k\to\infty} \ln|x+1| \Big|_1^k$

$= \lim_{k\to\infty} \left[\ln(k+1) - \ln(2)\right] = \infty - \ln(2) = \infty$

by integral test, since $\int_1^\infty \frac{1}{x+1}\,dx = \infty$ then $\sum_{n=1}^{\infty} \frac{1}{n+1} = \infty$

6. $\sum_{n=1}^{\infty} \frac{1}{n^{-3}}$

$p = -3 < 1 \Rightarrow$ diverges

7. $\sum_{n=1}^{\infty} \frac{3}{n^{3/2}}$

$$R_6 < \int_6^{\infty} \frac{3}{x^{3/2}}\,dx \rightarrow \lim_{b\to\infty} \int_6^{b} \frac{3}{x^{3/2}}\,dx \rightarrow \lim_{b\to\infty} -\frac{6}{\sqrt{x}}\Big|_1^b$$

$$< \lim_{b\to\infty} \left[-\frac{6}{\sqrt{b}} + \frac{6}{\sqrt{6}}\right] \rightarrow R_6 < \left(0 + \frac{6}{\sqrt{6}}\right) \approx 2.4495$$

8. $R_n < \int_n^{\infty} 4e^{-7x}\,dx$

$$R_n < \lim_{b\to\infty} \int_n^{b} 4e^{-7x}\,dx \rightarrow \lim_{b\to\infty} -\frac{4}{7e^{7x}}\Big|_n^b \rightarrow \lim_{b\to\infty} \left[-\frac{4}{7e^{7b}} + \frac{4}{7e^{7n}}\right]$$

$$R_n < 0 + \frac{4}{7e^{7n}}$$

$$\frac{4}{7e^{7n}} < 0.0001 \rightarrow 4 < 0.0007e^{7n} \rightarrow 5714.2857 < e^{7n}$$

$$\ln(5714.2857) < 7n \rightarrow \frac{\ln(5714.2857)}{7} \approx 1.236 < n$$

$n \geq 2$

9. $\sum_{n=2}^{\infty} \frac{1}{n-1}$

$b_n = \frac{1}{n-1} \geq \frac{1}{n} = a_n \geq 0$

$\geq \sum_{n=2}^{\infty} \frac{1}{n} = 0$

So, by the comparison theorem, since $\sum_{n=2}^{\infty} \frac{1}{n} = 0$, we know $\sum_{n=2}^{\infty} \frac{1}{n-1} = 0$

10. $\lim_{n\to\infty} \frac{\left(\frac{3n^4-11n^2+10}{12n^6-44n^4+40n^2}\right)}{\left(\frac{1}{8n^2}\right)}$

$= \lim_{n\to\infty} \frac{8n^2}{1} \cdot \frac{3n^4-11n^2+10}{12n^6-44n^4+40n^2} = \lim_{n\to\infty} \frac{24n^6-88n^4+80n^2}{12n^6-44n^4+40n^2}$

$= \lim_{n\to\infty} \frac{\cancel{12n^6}\left(2-\frac{22}{3n^2}+\frac{20}{3n^4}\right)}{\cancel{12n^6}\left(1-\frac{11}{3n^2}+\frac{10}{3n^4}\right)} = \lim_{n\to\infty} \frac{2-\cancel{\frac{22}{3n^2}}^{0}+\cancel{\frac{20}{3n^4}}^{0}}{1-\cancel{\frac{11}{3n^2}}^{0}+\cancel{\frac{10}{3n^4}}^{0}} = \frac{2-0+0}{1-0+0} = 2 \neq 0$

11. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ $\quad b_n = \frac{1}{n^2} \geq 0$

1) $b_{n+1} \leq b_n$

$\frac{1}{(n+1)^2} < \frac{1}{n^2}$ ✓

2) $\lim_{n\to\infty} \frac{1}{n^2} = 0$ ✓

So, by the alternating series test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges

12. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \quad p=2>1$

So, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent (and conditionally)

13. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$

$$b_n = \frac{(-1)^n}{n^2+1} \qquad b_{n+1} = \frac{(-1)^{n+1}}{(n+1)^2+1}$$

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{(n+1)^2+1}}{\frac{(-1)^n}{n^2+1}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{(-1)^n} \right|$$

$$= \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1} = \lim_{n\to\infty} \frac{n^2+1}{n^2+2n+2} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(1+\frac{1}{n^2}\right)}{\cancel{n^2}\left(1+\frac{2}{n}+\frac{2}{n^2}\right)} = \lim_{n\to\infty} \frac{1+\cancelto{0}{\frac{1}{n^2}}}{1+\cancelto{0}{\frac{2}{n}}+\cancelto{0}{\frac{2}{n^2}}} = 1 =: L$$

So, the ratio test fails, no information, try another test (if told to do so)

14. $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{\left(1-\frac{1}{n}\right)^n} = \lim_{n\to\infty} 1-\frac{1}{n} = 1-0 = 1 =: L$$

So, the root test fails, try some thing else

$$\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n \to \text{"}1^{\infty}\text{"} \qquad y = \left(1-\frac{1}{x}\right)^x$$

$$\ln(y) = \ln\left[\left(1-\frac{1}{x}\right)^x\right] \Rightarrow \lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} x\ln\left(1-\frac{1}{x}\right) \quad \text{"}\infty\cdot 0\text{"}$$

$$= \lim_{x\to\infty} \frac{\ln\left(1-\frac{1}{x}\right)}{1/x} = \frac{0}{0}$$

$$\stackrel{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{\left(1-\frac{1}{x}\right)} \cdot \cancel{\left[-\left(-\frac{1}{x^2}\right)\right]}}{\cancel{-\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{-1}{1-\frac{1}{x}} = \frac{-1}{1-0} = -1$$

$$\lim_{x\to\infty} \ln(y) = -1$$

$$\lim_{x\to\infty} e^{\ln(y)} = e^{-1}$$

$$\lim_{x\to\infty} y = e^{-1} \Rightarrow \lim_{x\to\infty} \left(1-\frac{1}{x}\right)^x = e^{-1} \neq 0$$

So, by the divergence test $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n$ diverges