Arc Length & Area of a Surface of a Revolution

- Functions of x: $\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$
- Functions of y: $\int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2+1}\,dy$
- Surface area of a revolution: $\int_a^b 2\pi f(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

1. $y=\frac{x^2}{2}-\frac{\ln x}{4}, 2 \le x \le 4$

$y' = x - \frac{1}{4x} \qquad \Delta x = \frac{2}{n} \qquad x_i = 2 + \frac{2}{n} i$

$$L = \int_2^4 \sqrt{1+\left(x-\frac{1}{4x}\right)^2}\,dx = \int_2^4 \sqrt{1+\left(\frac{4x^2-1}{4x}\right)^2}\,dx = \int_2^4 \sqrt{1+\frac{16x^4-8x^2+1}{16x^2}}\,dx$$

$$= \int_2^4 \frac{\sqrt{16x^4+8x^2+1}}{\sqrt{16x^2}}\,dx = \int_2^4 \frac{\sqrt{(4x^2+1)^2}}{4x}\,dx = \int_2^4 \frac{4x^2+1}{4x}\,dx = \int_2^4 \frac{4x^2}{4x}+\frac{1}{4x}\,dx$$

$$= \int_2^4 x+\frac{1}{4}\cdot\frac{1}{x}\,dx = \left[\frac{x^2}{2}+\frac{1}{4}\ln x\right]_2^4 = \left(\frac{4^2}{2}+\frac{1}{4}\ln(4)\right)-\left(\frac{2^2}{2}+\frac{1}{4}\ln(2)\right) = 6+\frac{1}{4}\ln(2)$$

2. $x = \ln[\cos(y)], -\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$

$\Delta y = \frac{2\pi}{4n}$ $\quad y_i = -\frac{\pi}{4} + \frac{2\pi}{4n} i$

$x' = \frac{1}{\cos(y)} \cdot (-\sin(y)) = -\tan(y)$

y	$x = \ln[\cos(y)]$
$-\frac{\pi}{4}$	$\ln[\cos(-\frac{\pi}{4})] = -\ln\sqrt{2} = -\frac{1}{2}\ln(2)$
0	$\ln[\cos(0)] = 0$
$\frac{\pi}{4}$	$\ln[\cos(\frac{\pi}{4})] = -\ln\sqrt{2}$

$$L = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{(-\tan(y))^2 + 1}\, dy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\tan^2(y) + 1}\, dy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\sec^2(y)}\, dy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec(y)\, dy$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec(y) \cdot \frac{[\sec(y) + \tan(y)]}{[\sec(y) + \tan(y)]}\, dy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2(y) + \sec(y)\tan(y)}{\sec(y) + \tan(y)}\, dy$$

$u = \sec(y) + \tan(y)$

$du = \sec(y)\tan(y) + \sec^2(y)\, dy$

$u(\frac{\pi}{4}) = \sqrt{2} + 1$

$u(-\frac{\pi}{4}) = \sqrt{2} - 1$

$$\int_{\sqrt{2}-1}^{\sqrt{2}+1} \frac{1}{u}\, du = \ln(u)\Big]_{\sqrt{2}-1}^{\sqrt{2}+1} = \ln(\sqrt{2}+1) - \ln(\sqrt{2}-1) = \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

3. $f(x) = \sqrt{4 - x^2}$, around the x-axis

$y = \sqrt{4 - x^2}$

$y^2 = \left(\sqrt{4 - x^2}\right)^2 = 4 - x^2$

$y^2 + x^2 = 4 = 2^2 \longrightarrow$

$(x-h)^2 + (y-k)^2 = r^2$

center $(h,k) = (0,0)$

$y' = \frac{-x}{\sqrt{4-x^2}}$

$r = 2$

$$S = \int_{-2}^{2} 2\pi\sqrt{4-x^2} \cdot \sqrt{1 + \frac{x^2}{4-x^2}}\, dx = \int_{-2}^{2} 2\pi \cancel{\sqrt{4-x^2}} \cdot \frac{\sqrt{4}}{\cancel{\sqrt{4-x^2}}}\, dx$$

$$= \int_{-2}^{2} 4\pi\, dx = 4\pi x \Big]_{-2}^{2} = 4\pi[2-(-2)] = 16\pi$$

4. $f(x) = 3x - 2$, $[2,4]$, around the y-axis

$y = 3x - 2 \longrightarrow x = \frac{y}{3} + \frac{2}{3}$ $\quad x' = \frac{1}{3}$

$$S = \int_{2}^{4} 2\pi\left(\frac{y}{3} + \frac{2}{3}\right)\sqrt{1 + \frac{1}{9}}\, dy = \int_{2}^{4} \frac{2\sqrt{10}\,\pi}{9}(y+2)\, dy$$

$$\frac{2\sqrt{10}\,\pi}{9}\int_{2}^{4}(y+2)\, dy = \frac{2\sqrt{10}\,\pi}{9}\left(\frac{y^2}{2} + 2y\right)\Big]_{2}^{4} = \frac{2\sqrt{10}\,\pi}{9}\left[(8+8)-(2+4)\right]$$

$$= \frac{2\sqrt{10}\,\pi}{9}(10) = \frac{20\sqrt{10}\,\pi}{9}$$