The Comparison & Limit Comparison Tests

- $n \geq N \sim\sim 0 < a_n < b_n$
  - $\sum_n a_n$ diverges, $\sum_n b_n$ diverges
  - $\sum_n b_n$ converges, $\sum_n a_n$ converges
- $a_n \geq 0; b_n \geq 0$
  - $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$, $\sum_n a_n$ & $\sum_n b_n$ both converge or diverge
    - $\lim_n \frac{a_n}{b_n} = 0$ & $\sum_n b_n$ converges, $\sum_n a_n$ converges
    - $\lim_n \frac{a_n}{b_n} = \infty$ & $\sum_n b_n$ diverge, $\sum_n a_n$ diverge
    - $\lim_n \frac{a_n}{b_n} = 0$ & $\sum_n b_n$ diverges, no info about $\sum_n a_n$
    - $\lim_n \frac{a_n}{b_n} = \infty$ & $\sum_n b_n$ converges, no info about $\sum_n a_n$

1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4+n^2+100}}$ $\quad 0 \leq \left(a_n = \frac{1}{\sqrt{n^4+n^2+100}}\right) \leq \left(\frac{1}{\sqrt{n^4}} = b_n\right)$

$$0 \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4}} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{p-series: } p=2} < \infty$$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4}} < \infty$ b/c it is a p-series w/ $p=2>1$

so, by the comparison theorem, since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4}} < \infty$,

then $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4+n^2+100}} < \infty$

2. $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ $\left(b_n = \frac{1}{2^n - 1}\right) \geq \left(\frac{1}{2^n} = a_n\right) > 0$

$\geq \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \ -1 < r < 1$

$\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{\frac{1}{2}}{1-\frac{1}{2}}, \ -1 < \frac{1}{2} < 1 \Rightarrow \frac{\frac{1}{2}}{\frac{1}{2}} = 1 < \infty$

So, $0 \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \leq \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

So, the comparison test fails, we gain NO information, and we try a different test

3. $\sum_{n=2}^{\infty} \frac{4n^2 + n}{\sqrt[3]{n^7 + n^3}}$ $\sqrt[3]{n^7 + n^3} \approx \sqrt[3]{n^7}$

$4n^2 + n \approx 4n^2$

$$\frac{4n^2+n}{\sqrt[3]{n^7+n^3}} \approx \frac{4n^2}{\sqrt[3]{n^7}} = \frac{4n^2}{n^{7/3}} = \frac{4}{n^{1/3}} \approx \frac{1}{n^{1/3}}$$

$\sum_{n=2}^{\infty} \frac{1}{n^{1/3}} = \infty$ b/c p-series: $p = \frac{1}{3} \leq 1$

$$\lim_{n\to\infty} \frac{\left(\frac{4n^2+n}{\sqrt[3]{n^7+n^3}}\right)}{\left(\frac{1}{n^{1/3}}\right)} = \lim_{n\to\infty} \frac{n^{1/3}}{1} \cdot \frac{4n^2+n}{\sqrt[3]{n^7+n^3}} = \lim_{n\to\infty} \frac{4n^{7/3}+n^{4/3}}{\sqrt[3]{n^7\left(1+\frac{n^3}{n^7}\right)}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^{7/3}}\left[4+\frac{n^{4/3}}{n^{7/3}}\right]}{\cancel{n^{7/3}}\sqrt[3]{1+\frac{1}{n^4}}} = \lim_{n\to\infty} \frac{4+\frac{1}{n}}{\sqrt[3]{1+\frac{1}{n^4}}} = \frac{4+0}{\sqrt[3]{1+0}} = 4 \neq 0$$

since $\lim_{n\to\infty} \frac{\left(\frac{4n^2+n}{\sqrt[3]{n^7+n^3}}\right)}{\left(\frac{1}{n^{1/3}}\right)} = 4 \neq 0$ & $\sum_{n=2}^{\infty} \frac{1}{n^{1/3}} = \infty$, then by the limit comparison test,

$\sum_{n=2}^{\infty} \frac{4n^2+n}{\sqrt[3]{n^7+n^3}} = \infty$

4. $\lim\limits_{n\to\infty} \dfrac{\left(\frac{1}{2^n-1}\right)}{\left(\frac{1}{2^n}\right)}$

$$\lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{\cancel{2^n}(1)}{\cancel{2^n}\left(1-\frac{1}{2^n}\right)} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}} = \frac{1}{1-0} = 1 \neq 0$$

Since $\lim\limits_{n\to\infty} \dfrac{\left(\frac{1}{2^n-1}\right)}{\left(\frac{1}{2^n}\right)} = 1 \neq 0$, and $\sum\limits_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty$, then by the limit comparison test,

$\sum\limits_{n=1}^{\infty} \frac{1}{2^n+1} < \infty$