Calculus with Parametric Curves

- Horizontal tangents: $\frac{dy}{dt} = 0$, when $\frac{dx}{dt} \neq 0$
- Vertical tangents: $\frac{dx}{dt} = 0$, when $\frac{dy}{dt} \neq 0$
- Multiple tangents may occur at a point

1. $x(t) = 4\cos(t)$, $y(t) = 3\sin(t)$, $[0, 2\pi]$, find $\frac{dy}{dx}$ and the equation for the slope of the tangent line.

A) $x(t) = 4\cos(t) \quad y(t) = 3\sin(t) \quad [0, 2\pi]$

$x'(t) = -4\sin(t) \quad y'(t) = 3\cos(t)$

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3\cos(t)}{-4\sin(t)} = -\frac{3}{4}\cot(t)$

B) $m = \left.\frac{dy}{dt}\right|_{t=\frac{\pi}{6}} = -\frac{3}{4}\cot\left(\frac{\pi}{6}\right) = -\frac{3}{4}\sqrt{3} = -\frac{3\sqrt{3}}{4}$

$x\left(\frac{\pi}{6}\right) = 4\cos\left(\frac{\pi}{6}\right) = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$

$y\left(\frac{\pi}{6}\right) = 3\sin\left(\frac{\pi}{6}\right) = 3\left(\frac{1}{2}\right) = \frac{3}{2}$

$\left(2\sqrt{3}, \frac{3}{2}\right)$

$y - y_1 = m(x - x_1) \rightarrow$ point slope

$y - \frac{3}{2} = -\frac{3\sqrt{3}}{4}(x - 2\sqrt{3}) = -\frac{3\sqrt{3}}{4}x + \frac{3\sqrt{3}}{\cancel{4}_2} \cdot \cancel{2}\sqrt{3}$

$y - \frac{3}{2} = -\frac{3\sqrt{3}}{4}x + \frac{9}{2} \Rightarrow y = -\frac{3\sqrt{3}}{4}x + 6$

2. $x(t) = t^3 - 3t$, $y(t) = 3t^2 - 9$, find t=0 and t=1, find the tangents.

$x(t) = t^3 - 3t$ & $y(t) = 3t^2 - 9$

$t = 0 \rightarrow x(0) = 0 \quad y(0) = -9$

$t = 1 \rightarrow x(1) = -2 \quad y(1) = -6$

A) $\frac{dy}{dx} = \frac{6t}{3t^2 - 3} = \frac{6t}{3(t^2 - 1)} = \frac{2t}{(t+1)(t-1)}$

$\frac{dy}{dx}$ is undefined @ $t = -1, 1$

$t = 1$: $x(1) = -2$, $y(1) = -6$ $\quad (-2, -6)$

$t = -1$: $x(-1) = 2$, $y(-1) = -6$ $\quad (2, -6)$

$\left.\begin{matrix}(-2,-6)\\(2,-6)\end{matrix}\right\}$ vertical tangents

B) $\frac{dy}{dx} = 0 = \frac{2t}{(t-1)(t+1)} \rightarrow 2t = 0 \rightarrow t = 0$

$t = 0$: $x(0) = 0$, $y(0) = -9$ $\quad (0, -9) \rightarrow$ horizontal tangent

C) $x(t) = t^3 - 3t = 0 \rightarrow t(t^2 - 3) = 0 \rightarrow \cancel{t = 0}$ & $\pm\sqrt{3}$

$y(t) = 3t^2 - 9 = 0 \rightarrow t^2 = \frac{9}{3} \rightarrow t = \pm\sqrt{3}$

$(0, 0) \rightarrow t = -\sqrt{3}$ & $\sqrt{3}$ $\qquad \frac{dy}{dx} = \frac{2t}{t^2 - 1}$

$t = -\sqrt{3}$: $m = \frac{2(-\sqrt{3})}{(-\sqrt{3})^2 - 1} = \frac{-2\sqrt{3}}{3 - 1} = \frac{-2\sqrt{3}}{2}$ $\qquad m = -\sqrt{3}$

$t = \sqrt{3}$: $m = \frac{2\sqrt{3}}{\sqrt{3}^2 - 1} = \frac{2\sqrt{3}}{2} = +\sqrt{3}$ $\qquad (0, 0)$

$y = mx + b$

$y = \sqrt{3}x \qquad y = -\sqrt{3}x$