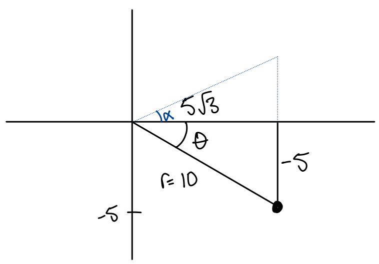
# Polar Coordinates & Areas and Lengths in Polar Coordinates

- $r^2 = x^2 + y^2$
- $\theta = \tan^{-1}\left(\frac{x}{y}\right)$
- $x = r\cos\theta, y = r\sin\theta$
- $A = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{2}[f(\theta_i)]^2 \Delta\theta = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2 d\theta = \frac{1}{2}\int_\alpha^\beta r^2 d\theta$
- $L = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$
- $\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$

1. $(5\sqrt{3}, -5)$ to polar coordinates




$r = x^2 + y^2$

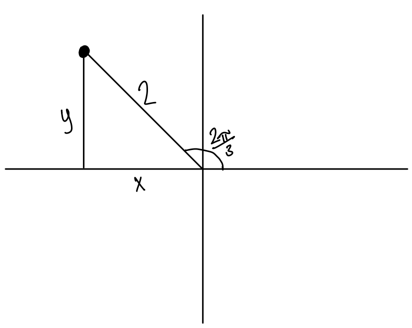
$r = \sqrt{(5\sqrt{3})^2 + (-5)^2} = 10$

$\alpha = \tan^{-1}\left(\frac{-5}{5\sqrt{3}}\right) = \frac{\pi}{6}$

$\theta = -\frac{\pi}{6}$

$\left(10, -\frac{\pi}{6}\right)$ or $\left(10, \frac{5\pi}{3}\right)$

2. $\left(2, \frac{2\pi}{3}\right)$ to rectangular coordinates




$x = r\cos\theta$

$= 2\cos\frac{2\pi}{3}$

$= -1$

$y = r\sin\theta$

$= 2\sin\frac{2\pi}{3}$

$= \sqrt{3}$

$(-1, \sqrt{3})$

3. $r = \cos(3\theta), \left[0, \frac{2\pi}{3}\right]$, find the area

$$\int_0^{\pi/2} \frac{1}{2}\left[\cos(3\theta)\right]^2 d\theta = \int_0^{\pi/2} \left[\frac{1+\cos(6\theta)}{2}\right] d\theta = \frac{1}{2}\left[\theta + \frac{\sin(6\theta)}{6}\right]_0^{\pi/2}$$

$$\frac{1}{2}\left[\left(\frac{\pi}{2} + \underbrace{\frac{1}{6}\sin(3\pi)}_{0}\right) - \left(\underbrace{0 + \frac{1}{6}\sin(0)}_{0}\right)\right] = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

4. $r = 2, r = 4\cos(2\theta), [0, 4\pi]$, find the area

$4\cos(2\theta) = 2$

$\cos(2\theta) = \frac{1}{2}$

$\alpha = \cos^{-1}\left(+\frac{1}{2}\right) = \frac{\pi}{3}$

$2\theta_1 = \alpha = \frac{\pi}{3}$

$2\theta_2 = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

$2\theta_3 = \frac{7\pi}{3}$

$2\theta_4 = \frac{11\pi}{3}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

$4\cos(2\theta) = -2$

$\cos(2\theta) = -\frac{1}{2}$

$\alpha = \cos^{-1}\left(+\frac{1}{2}\right) = \frac{\pi}{3}$

$2\theta_5 = \pi - \alpha = \frac{2\pi}{3}$

$2\theta_6 = \pi + \alpha = \frac{4\pi}{3}$

$2\theta_7 = \frac{8\pi}{3}$

$2\theta_8 = \frac{10\pi}{3}$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

$$8\int_0^{\pi/6}\left[\frac{1}{2}(4\cos(2\theta))^2 - \frac{1}{2}(2)^2\right] d\theta = 4\int_0^{\pi/6} 16\cos^2(2\theta) - 4\, d\theta = 4\int_0^{\pi/6} 16\,\frac{1+\cos(4\theta)}{2} - 4\, d\theta$$

$$= 4\int_0^{\pi/6} 8(1+\cos(4\theta)) - 4\, d\theta = 4\int_0^{\pi/6} 4 + 8\cos(4\theta)\, d\theta = 4\left[4\theta + \frac{8\sin(4\theta)}{4}\right]_0^{\pi/6} = 4\left\{\left[4\left(\frac{\pi}{6}\right) + 2\sin\left(\frac{2\pi}{3}\right)\right] - \left[0 + 2\sin(0)\right]\right\}$$

$$= 4\left[\frac{2\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right)\right] = \frac{8\pi}{3} + 4\sqrt{3}$$

5. $r = \theta^2 - 1, \theta = [1,2]$, find the arc length

$f(\theta) = \theta^2 - 1 \quad f'(\theta) = 2\theta$

$$L = \int_1^2 \sqrt{(\theta^2-1)^2 + (2\theta)^2}\, d\theta = \int_1^2 \sqrt{(\theta^4 - 2\theta^2 + 1) + 4\theta^2}\, d\theta = \int_0^2 \sqrt{\theta^4 + 2\theta^2 + 1}\, d\theta$$

$$= \int_1^2 \sqrt{(\theta^2+1)^2}\, d\theta = \int_1^2 \theta^2 + 1\, d\theta = \left.\frac{\theta^3}{3} + \theta\right]_1^2 = \left(\frac{2^3}{3} + 2\right) - \left(\frac{1}{3} + 1\right) = \frac{10}{3}$$

6. $r = 8 \ln\theta, \theta = 6e$, find the arc length

$r' = \frac{8}{\theta}$

$$\frac{dy}{dx} = \frac{\frac{8}{\theta}\sin\theta + 8\ln\theta\cos\theta}{\frac{8}{\theta}\cos\theta - 8\ln\theta\sin\theta} = \frac{\frac{8}{6e}\sin(6e) + 8\ln(6e)\cos(6e)}{\frac{8}{6e}\cos(6e) - 8\ln(6e)\sin(6e)} \approx -1.53$$