Power Series & Representation of Functions as Power Series

- $\sum_{n=0}^{\infty} c_n(x-a)^n$ satisfies exactly one of the following
  - The series converges at $x = a$ and diverges for all $x \neq a$
  - The series converges for all real numbers x
  - There exists a real number $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$
    - At the values x where $|x-a| = R$, the series may converge or diverge

1. $|x-a| < R \rightarrow a - R < x < a + R$
2. $|x-a| > R \rightarrow x - a > R$ or $x - a < -R$
3. $|x-a| = R \rightarrow x - a = R$ or $x - a = -R$

- If the power series converges for a single number, then $R = 0$
- If the power series converges for all real numbers, then $R = \infty$
- The ratio test is handy for finding the interval $|x-a| < R$

$a=0$ (centered about 0)

1. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| =: L < 1$ (converges if true)

$$\lim_{n\to\infty} \left|\frac{\left(\frac{x^{n+1}}{(n+1)!}\right)}{\left(\frac{x^n}{n!}\right)}\right| = \lim_{n\to\infty} \left|\frac{x^{\cancel{n+1}}}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{\cancel{x^n}}\right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = 0 =: L < 1 \rightarrow \text{converges}$$

converges for all real numbers $(-\infty, \infty)$ $R = \infty$

2. $\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$ $a=0$

$$\lim_{n\to\infty}\left|\frac{\left(\frac{10^{n+1}x^{n+1}}{(n+1)^3}\right)}{\frac{10^n x^n}{n^3}}\right| = \lim_{n\to\infty}\left|\frac{10^{\cancel{n+1}}x^{\cancel{n+1}}}{(n+1)^3}\cdot\frac{n^3}{\cancel{10^n}\cancel{x^n}}\right| = \lim_{n\to\infty}\left|10x\left(\frac{n}{n+1}\right)^3\right|$$

$= |10x|\cdot\frac{1}{1} \to$ assume $L<1$ $\quad |10x|<1 \to |x|<\frac{1}{10} =: R \to -\frac{1}{10}<x<\frac{1}{10}$

$x=\frac{1}{10}$ $\quad \sum_{n=1}^{\infty}\frac{10^n(\frac{1}{10})^n}{n^3} = \sum_{n=1}^{\infty}\frac{1}{n^3} \to p=3>1 \to$ converges

$x=-\frac{1}{10}$ $\quad \sum_{n=1}^{\infty}\frac{10^n(-\frac{1}{10})^n}{n^3} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} \to$ absolute convergence: $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^3}\right| = \sum_{n=1}^{\infty}\frac{1}{n^3}$

max interval of convergence is $\left[-\frac{1}{10}, \frac{1}{10}\right]$

3. $f(x) = \frac{7}{7x+1}$; find the interval of convergence

$f(x) = \frac{7}{1-(-7x)}$ $\quad a=7 \quad r=-7x$

$|-7x|<1 \to |x|<\frac{1}{7}$ $\quad \left(-\frac{1}{7}, \frac{1}{7}\right)$

4. $f(x) = \frac{x^6}{9x^5-2}$; turn into $\sum_{n=0}^{\infty} \ldots$

$$f(x) = \frac{x^6}{-2-(-9x^5)} = \frac{-\frac{x^6}{2}}{1-\left(\frac{9x^5}{2}\right)} \qquad a = -\frac{x^6}{2} \qquad r = \frac{9x^5}{2}$$

$$\sum_{n=0}^{\infty}\left(-\frac{x^6}{2}\right)\left(\frac{9x^5}{2}\right)^n = \sum_{n=0}^{\infty} -\frac{1}{2}\left(\frac{9}{2}\right)^n x^{5n+6}$$