Absolute Convergence and The Ratio & Root Tests

- Absolute convergence:
  - If $\sum_n |a_n|$ converges, then we say the series $\sum_n a_n$ is absolutely convergent
  - If $\sum_n |a_n|$ diverges but $\sum_n a_n$ converges, then we say the series $\sum_n a_n$ is conditionally convergent
  - If $\sum_n |a_n|$ is absolutely convergent, then $\sum_n a_n$ is conditionally convergent
- Ratio test:
  - $\sum_n a_n$, let $L = \lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right|$
    - If $L < 1$, then the series is absolutely convergent
    - If $L > 1$ or $L = \infty$, then the series is divergent
    - If $L = 1$, then the test fails and we gain no information
- Root test:
  - $\sum_n a_n$, let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$
    - If $L < 1$, then the series is absolutely convergent
    - If $L > 1$ or $L = \infty$, then the series is divergent
    - If $L = 1$, then the test fails and we gain no information

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ $\rightarrow$ conditionally convergent

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

NOT absolutely convergent

2. $\sum_{n=1}^{\infty} \frac{1}{n!}$ $\rightarrow \lim_{n\to\infty} \left| \frac{\left(\frac{1}{(n+1)!}\right)}{\left(\frac{1}{n!}\right)} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{\cancel{n!}}{(n+1)\cancel{n!}}$

$\lim_{n\to\infty} \frac{1}{n+1} = 0 =: L < 1$

so, by the ratio test, $\sum_{n=1}^{\infty} \frac{1}{n!}$ is absolutely convergent

3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^n}$ $\rightarrow \lim_{n\to\infty} \sqrt[n]{\left| \frac{(-1)^n}{n^n} \right|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^n}} = \lim_{n\to\infty} \frac{1}{n} = 0 =: L < 1$

by the root test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^n}$ is absolutely convergent