Test Prep 4 Worksheet 2

1. $f^{(n)}(8) = \frac{(-1)^n n!}{5^n(n+1)}$; f is centered at 8, find the Taylor series, find the radius of convergence

$$\sum_{n=0}^{\infty} \frac{f^n(8)}{n!}(x-8)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{n!}}{5^n(n+1)\cancel{n!}}(x-8)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n(n+1)}(x-8)^n$$

$$\lim_{n\to\infty}\left|\frac{(-1)^{\cancel{n+1}}(x-8)^{\cancel{n+1}}}{5^{\cancel{n+1}}(n+2)} \cdot \frac{\cancel{5^n}(n+1)}{\cancel{(-1)^n}\cancel{(x-8)^n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)(x-8)(n+1)}{5(n+2)}\right| \to \frac{1}{5}|x-8| \lim_{n\to\infty}\frac{n+1}{n+2}$$

$$\frac{1}{5}|x-8| < 1 \Rightarrow |x-8| < 5 \to R=5$$

2. $f(x) = \frac{4}{1+x}$; a=2, use the Taylor series to find the first four nonzero terms

n	$f^n(x)$	$f^n(2)$
0	$\frac{4}{1+x}$	$\frac{4}{3}$
1	$-\frac{4}{(1+x)^2}$	$-\frac{4}{9}$
2	$\frac{8}{(1+x)^3}$	$\frac{8}{27}$
3	$-\frac{24}{(1+x)^4}$	$-\frac{24}{81}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= \frac{4}{3} + \frac{-4/9}{1!}(x-2) + \frac{8/27}{2!}(x-2)^2 + \frac{-24/81}{3!}(x-2)^3$$

$$= \frac{4}{3} - \frac{4}{9}(x-2) + \frac{4}{24}(x-2)^2 - \frac{4}{81}(x-2)^3$$

3. $f(x) = \cos x$; find the Maclaurin series for f(x) using the definition of a Maclaurin series, find the radius of convergence

n	$f^n(x)$	$f^n(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1
$\vdots$	$\vdots$	$\vdots$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\lim_{n\to\infty} \left| \frac{x^{\cancel{2n}+2}}{(2n+2)!} \cdot \frac{(2n)!}{\cancel{x^{2n}}} \right| = \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+1)} = 0 < 1$$

so, $R = \infty$

4. $f(x) = x^5 + 2x^3 + x$; a=3, find the Taylor series for f(x) at the given value of a, find the radius of convergence

n	$f^n(x)$	$f^n(3)$
0	x^5+2x^3+x	300
1	$5x^4+6x^2+1$	460
2	$20x^3+12x$	576
3	$60x^2+12$	552
4	$120x$	360
5	120	120
6	0	0
$\vdots$	$\vdots$	$\vdots$

$$= 300 + \frac{460}{1!}(x-3) + \frac{576}{2!}(x-3)^2 + \frac{552}{3!}(x-3)^3 + \frac{360}{4!}(x-3)^4 + \frac{120}{5!}(x-3)^5$$

$$= 300 + 460(x-3) + 288(x-3)^2 + 92(x-3)^3 + 15(x-3)^4 + (x-3)^5$$

converges for all x, $R = \infty$

5. $f(x) = e^{-4x}\sin 2x$; a=0, find the Taylor polynomial $T_3(x)$

$T_3(x)$

n	$f^n(x)$	$f^n(0)$
0	$e^{-4x}\sin(2x)$	0
1	$e^{-4x}(2\cos(2x) - 4\sin(2x))$	2
2	$e^{-4x}(12\sin(2x) - 16\cos(2x))$	-16
3	$e^{-4x}(88\cos(2x) - 16\sin(2x))$	88

$$T_3(x) = \sum_{n=0}^{3} \frac{f^n(0)}{n!} x^n = 2x - 8x^2 + \frac{44}{3}x^3$$

6. $f(x) = \frac{1}{x}$; a=1, n=2, $0.7 \le x \le 1.3$, find the Taylor polynomial $T_2(x)$, use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval

$\frac{1}{x} \approx T_2(x)$

n	$f^n(x)$	$f^n(1)$
0	$\frac{1}{x}$	1
1	$-\frac{1}{x^2}$	-1
2	$\frac{2}{x^3}$	3

$$1 - \frac{1}{1!}(x-1) + \frac{2}{2!}(x-1)^2 = 1 - (x-1) + (x-1)^2$$

$|R_2(x)| \le \frac{M}{3!}|x-1|^3$ $\quad M \ge |f'''(x)|$

$f'''(x) = -\frac{6}{x^3}$

$0.7 \le x \le 1.3$

$|x-1| \le 0.3 \rightarrow |x-1|^3 \le 0.027$

since $|f'''(x)|$ is decreasing on $[0.7, 1.3]$, we can take $M = |f'''(0.7)| = \frac{6}{0.7^4}$

so $|R_2(x)| \le \frac{6/(0.7)^4}{6}(0.027) \approx 0.1125$

7. $\sin x \approx x - \frac{x^3}{6}$; $|error| < 0.01$, use the alternating series estimation theorem or Taylor's Inequality to estimate the range of values of x for which the given approximation is accurate to within the stated error

$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots$

$\left|\frac{1}{5!}x^5\right| < 0.01 \rightarrow |x^5| < 120(0.01) \rightarrow |x| < (1.2)^{1/5} \approx 1.037$

8. A car is moving with speed 60 m/s and acceleration 6 m/[2] at a given instant. Using a second-degree Taylor polynomial, estimate how far the car moves in the next second.

$v(t) = s'(t)$

$a(t) = s''(t)$

$T_2(t) = S(0) + V(0)t + \frac{a(0)}{2}t^2$

$60t + 3t^2$

$@\ t=1 \rightarrow S(1) \approx T_2(1)$

$60(1) + 3(1)^2 = 63m$

a. Will it be reasonable to use this polynomial to estimate the distance traveled during the next minute?
   i. Yes
   ii. No