Taylor and Maclaurin Series

- $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$; for $|x-a| < R$
  - $f(x) = c_0 + c_1(x-a)^1 + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$
  - $f(a) = c_0 \rightarrow c_0 = \frac{f(a)}{1} = \frac{f(a)}{0!}$
    - $c_n = \frac{f^n(a)}{n!}, n = 0,1,2, \ldots$
- Taylor series: x=a
  - If a function f has a power series at a that converges to f on some open interval containing a, then that power series is the Taylor series for f at a
- $\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$
- Maclaurin series: x=0
  - The n^{th} Taylor polynomial for f at $a = 0$ is known as the n^{th} Maclaurin polynomial for f
  - The remainder term $R_n(x)$ is $R_n(x) = f(x) - p_n(x)$

1. $f(x) = \frac{1}{x}$; Taylor series

$= \sum_{n=0}^{\infty} c_n (x-1)^n$

$f'(x) = -x^{-2}$ $\quad f''(x) = 2x^{-3}$ $\quad f'''(x) = -6x^{-4}$ $\quad f^4(x) = 24x^{-5}$

$f'(1) = -1!$ $\quad f''(1) = 2!$ $\quad f'''(1) = -3!$ $\quad f^4(1) = 4!$

$f^n(1) = (-1)^n n!$

Taylor series for $\frac{1}{x}$ @ $a=1$ is $\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cancel{n!}}{\cancel{n!}}(x-1)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{n=0}^{\infty} (1-x)^n = \sum_{n=0}^{\infty} \left[(-1)(-1+x)\right]^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n, \quad |1-x| < 1 \Rightarrow 0 < x < 2$$

So, $\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$, $0 < x < 2$

2. $f(x) = \frac{1}{x}$; Maclaurin polynomials

$a = 1$

$f'(x) = -x^{-2}$ $\quad f(1) = 1$

$f''(x) = 2x^{-3}$ $\quad f'(1) = -1$

$f'''(x) = -3 \cdot 2x^{-4}$ $\quad f''(1) = 2$

$f^{4}(x) = 4 \cdot 3 \cdot 2x^{-5}$ $\quad f'''(1) = -3!$

$f^{5}(x) = -5 \cdot 4 \cdot 3 \cdot 2x^{-6}$ $\quad f^{4}(1) = 4!$

$f^{6}(x) = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2x^{-7} = \frac{6!}{x^7}$ $\quad f^{5}(1) = -5!$

$$p_5(x) = \sum_{k=0}^{5} \frac{f^k(1)}{k!}(x-1)^k$$

$$= \sum_{k=0}^{5} \frac{(-1)^k \cancel{k!}}{\cancel{k!}}(x-1)^k$$

$$= \sum_{k=0}^{5} (-1)^k (x-1)^k$$

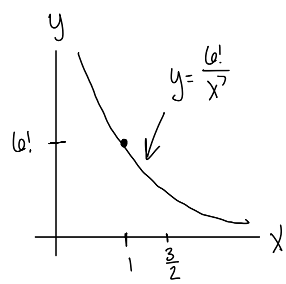
$|R_n(x)| \le \frac{m|x-a|^{n+1}}{(n+1)!}$, $|f^{n+1}(x)| \le m$ for all x in I

$n=5$ $a=1$ $x=\frac{3}{2}$ $I=[1, \frac{3}{2}]$

m = max y value on interval $= 6! = 720$

$|R_5(x)| \le \frac{\cancel{6!}|x-1|^6}{\cancel{6!}}$

$|R_5(\frac{3}{2})| \le |\frac{3}{2}-1|^6 = \frac{1}{2^6} \approx 0.0156$



3. $(2x-3)^4$; Binomial series

$(2x-3)^4 = ?$ $\quad r=4 \quad a=2x \quad b=-3$

$= \binom{4}{0}(2x)^4(-3)^0 + \binom{4}{1}(2x)^3(-3)^1 + \binom{4}{2}(2x)^2(-3)^2 + \binom{4}{3}(2x)^1(-3)^3 + \binom{4}{4}(2x)^0(-3)^4$

$= 16x^4 + 4(8x^3)(-3) + 6(4x^2)(9) + 4(2x)(-27) + 81$

$= 16x^4 - 96x^3 + 216x^2 - 216x + 81$

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \dots \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}x^n + \dots$$

where $\binom{r}{n} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}$ for $|x|<1$

4. A) $\cos(\sqrt{x})$, B) $\cosh(x)$; Maclaurin series expansion

A) $\cos\sqrt{x}$ $\quad\quad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, any x

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{x})^{2n}}{(2n)!}, \quad x \geq 0$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x)^n}{(2n)!}, \quad x \geq 0$$

B) $\cosh(x) = \frac{e^x + e^{-x}}{2}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ any } x$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}, \text{ any } x$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x)^n}{n!}$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{(x^n + (-1)^n x^n)}{n!}\right]$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} [1 + (-1)^n] \frac{x^n}{n!}$$

$$= \frac{1}{2}\left[(1+1)\frac{x^0}{0!} + (1-1)\frac{x^1}{1!} + (1+1)\frac{x^2}{2!} + (1-1)\frac{x^3}{3!} + \dots\right]$$

$$= \frac{1}{2}\left[\overset{k=0}{\frac{2x^0}{0!}} + 0 + \overset{k=1}{\frac{2x^2}{2!}} + 0 + \overset{k=2}{\frac{2x^4}{4!}} + \dots\right]$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty} \frac{2x^{2k}}{(2k)!}\right] = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \text{ all real \#s}$$

$$\int_0^1 \cos(\sqrt{x})\,dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}\right] dx = \sum_{n=0}^{\infty} \left[\int_0^1 \frac{(-1)^n x^n}{(2n)!}\,dx\right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{n+1}}{(2n)!(n+1)}\Bigg|_0^1\right] = \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{(2n)!(n+1)}(1^{n+1} - 0^{n+1})\right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(n+1)} \leftarrow \text{alternating series}$$

$|R_N| \leq b_{n+1} < 0.01$

$b_n = \frac{1}{(2n)!(n+1)}$ $\quad$ 1) $\lim_{n\to\infty} \frac{1}{(2n)!(n+1)} = 0$ $\quad$ 2) $b_{n+1} < b_n$ ✓

$$\frac{1}{[2(N+1)]![(N+1)+1]} = \frac{1}{(2N+2)!(N+2)} < 0.01$$

$N=1$: $\frac{1}{4!(3)} = \frac{1}{72}$

$N=2$: $\frac{1}{6!(4)} = \frac{1}{2880}$

$$\int_0^1 \cos(\sqrt{x})\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(n+1)} \approx \sum_{n=0}^{2} \frac{(-1)^n}{(2n)!(n+1)} = \frac{1}{0!(1)} - \frac{1}{2!(2)} + \frac{1}{4!(3)}$$

$$= 1 - \frac{1}{4} + \frac{1}{72} = \frac{55}{72} \approx 0.7639$$